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A Basis for Polynomial Solutions to Systems of Linear Constant Coefficient PDE's

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$N_{\leqslant M} = \{u(x) \in K[x]_{\leqslant M} \mid P_k(\partial/\partial x_1, ..., \partial/\partial x_n)\ u(x) = 0;\ k = 1, 2, ..., r\}$ be the set of all polynomial solutions (of degree $\leqslant M$) to this system of partial differential equations. We solve the problem of finding an easily computed basis for the vector space $N_{\leqslant M}$. To do this we use a certain associative, and commutative algebra (defined over K), namely $K[\beta] = K[\beta_1, \beta_2, ..., \beta_n]$ where $\{P_k(\beta) = 0 \mid k = 1, ..., r\}$ and $\{\beta_1^{m_1} \beta_2^{m_2} \cdots \beta_n^{m_n} = 0 \mid m_1 + \cdots + m_n = M + 1\}$. Let the vector space $K[\beta]_{\leqslant M}$ equal the span over K of $\{\beta_1^{m_1} \beta_2^{m_2} \cdots \beta_n^{m_n} \mid m_1 + \cdots + m_n \leqslant M\}$. We show how the expression $\sum_{j=0}^{M} (x_1 \otimes \beta_1 + \cdots + x_n \otimes \beta_n)^j / j!$ can be used to find an easily computed basis for $N_{\leqslant M}$.

INTRODUCTION

Let P_k, $k = 1, 2, ..., r$ be polynomials in n variables (with coefficients from K) and let $K[x]_{\leqslant M}$ represent the set of all polynomials of degree less than or equal M in $x = (x_1, x_2, ..., x_n)$ (throughout this paper K will represent either the complex or the real numbers). We will find an easily computed basis for the vector space

$$N_{\leqslant M} = \left\{ u \in K[x]_{\leqslant M} \,\middle|\, P_k\left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right) u = 0;\ k = 1, 2, ..., r \right\}. \tag{1}$$

As the introduction in [1] points out, the problem of finding a basis for the vector space (1) is both natural and basic. [1] also points out that little has been done in addressing the general problem. In that same paper a method is discussed for finding a basis for $N_{\leqslant M}$ when $\{w \in C^n \mid P_k(w), k = 1, ..., r\}$ is smooth and irreducible. In [2] and [3] a method was

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developed for finding a basis in a large number of cases where the P_k are of homogeneous order. In discussing a closely related question, Stiller and, in earlier work, Michelli established various facts regarding the dimension of $N_{\leqslant M}$ when the P_k are also of homogeneous order (see [4]). There have also been numerous publications discussing polynomial solutions for special cases of (1). For example, much has been written about polynomial solutions to Laplace's equation in several variables (see any book discussing spherical harmonics; see e.g., [11]). The method presented in this paper extends the method appearing in [2] and [3] and is distinct from the methods presented by other authors. Furthermore, our method is simple and will give a basis for all cases of (1).

It is well know that the real and imaginary parts of polynomials in $z = x_1 + x_2 i$ form a basis for the set of all polynomial solutions to Laplace's equation $(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2)\ u(x_1, x_2) = 0$. We will be guided by the similarity between the algebraic relationship $1^2 + i^2 = 0$ (which relates the elements which form a basis for the complex number system) and the form of the 2 dimensional Laplacean operator. Specifically, to find all polynomial solutions in (1), we use a finitely generated, commutative, and associative algebra $K[\beta] = K[\beta_1, \beta_2, ..., \beta_n]$ whose generators satisfy the system of algebraic equations

$$\{P_k(\beta) = 0 \mid k = 1, 2, ..., r\}$$

and

$$\{\beta_1^{m_1} \beta_2^{m_2} \cdots \beta_n^{m_n} = 0 \mid m_1 + \cdots + m_n = M + 1\} \tag{2}$$

We show how the expression $\sum_{j=1}^{M} (\sum_{i=1}^{n} x_i \otimes \beta_i)^j / j!$ can be used to find a basis for $N_{\leqslant M}$.

Notation. We let $\mathbf{N}$ be the set of non-negative integers and we let C represent the complex numbers. A symbol w will generally represent the ordered n-tuple $(w_1, w_2, ..., w_n)$ and $\{w\}$ will represent the set $\{w_1, w_2, ..., w_n\}$. As is usual, $K[w]$ will represent the set of all polynomials in $w_1, w_2, ..., w_n$. For M in $\mathbf{N}$, $K[w]_M$ will equal the vector space spanned by all polynomials of degree M and $K[w]_{\leqslant M}$ will equal the vector space spanned by all polynomials degree less than or equal M. We use the standard notation of letting w^m represent $w_1^{m_1}\ w_2^{m_2} \cdots w_n^{m_n}$ where $m = (m_1, m_2, ..., m_n)$ is in $\mathbf{N}^n$. For m in $\mathbf{N}^n$ we define $|m| = m_1 + m_2 + \cdots + m_n$ and let $\mathbf{N}^n_{\leqslant M} = \{m \in \mathbf{N}^n \mid |m| \leqslant M\}$, $X^m = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} / m_1! m_2! \cdots m_n!$ and $D^m = D_1^{m_1} \cdots D_n^{m_n} = \partial^{m_1 + \cdots + m_n} / \partial x_1^{m_1} \cdots \partial x_n^{m_n}$. Also, for m in $\mathbf{Z}^n - \mathbf{N}^n$ we set $X^m = 0$. So, for example, for any i, j in $\mathbf{N}^n$ we have $D^i X^j = X^{j-i}$.

A Pairing of Vector Spaces

General references for the material in this section are [5, pp. 62–81] and [6, p. 139] and [1, p. 558].

The function $\langle\ ,\ \rangle$: $C[x]\times C[D]\to C$ defined by $\langle f(x), P(D)\rangle = \bar{P}(D)$ $f(x)|_0$ (we apply the complex conjugate of the operator $P(D)$ to $f(x)$ and then evaluate the result at 0) is bilinear. Since $D^j X^j = 1$ we see that $\langle\ ,\ \rangle$ is non-degenerate. Also since $\langle X^i, D^j\rangle = \delta_{ij}$, $\{D^j\}$ and $\{X^j\}$ for j in $\mathbf{N}^n$ are dual bases.

Let $C[D]^* = \mathrm{Hom}(C[D], C)$. Since $\langle\ ,\ \rangle$ is non-degenerate there is a linear isomorphism Φ: $C[D]^*\to C[x]$ given by the following conditions: For $\Psi\in C[D]^*$ we have

$$\Phi(\Psi) = f \Leftrightarrow \langle f, Q\rangle = \Psi(Q) \qquad \text{for all} \quad Q\in C[D].$$

Let $P_1(D), ..., P_r(D)$ be polynomials in $C[D]$ of any degrees. We extend this set of polynomials to $P_1(D), ..., P_r(D), ..., P_v(D)$, by joining the elements of $\{D^m \mid |m| = M+1\}$. Let $J = \langle P_1(D), ..., P_v(D)\rangle \subset C[D]$ (the ideal generated by $P_1(D), ..., P_v(D)$) and let $J_{\leqslant M} = J\cap C[D]_{\leqslant M}$. We also let $N = \{f(x)\in C[x] \mid \langle f(x), P_i(D)\rangle = 0$ for $i = 1, 2, ..., v\}$ and $N_{\leqslant M} = N\cap C[x]_{\leqslant M}$ (it is easy to verify that this is the same $N_{\leqslant M}$ that we defined in (1)). $J_{\leqslant M}$ and $N_{\leqslant M}$ are, of course, vector spaces. The restriction $\langle\ ,\ \rangle_{\leqslant M}$ of $\langle\ ,\ \rangle$ to $C[x]_{\leqslant M}\times C[D]_{\leqslant M}$ is still bilinear and non-degenerate. Therefore $\Phi = C[D]^*_{\leqslant M}\to C[x]_{\leqslant M}$ is given by $\Phi(\Psi) = f\Leftrightarrow \langle f, Q\rangle_{\leqslant M} = \Psi(Q)$ for all $Q\in C[D]_{\leqslant M}$.

Lemma 1. $N_{\leqslant M}\cong J^{\perp}_{\leqslant M}$.

Proof. Choose a basis

$$Q_1(D), ..., Q_t(D) \text{ for } C[D]_{\leqslant M} \text{ so that } Q_{s+1}(D), ..., Q_t(D)$$
$$\text{is a basis for } J_{\leqslant M}. \tag{3}$$

Therefore $N_{\leqslant M} = \{f(x)\in C[x]_{\leqslant M} \mid \langle f(x), Q_i(D)\rangle_{\leqslant M} = 0 \text{ for } i = s+1, ..., t\}$. Let $\Psi_1, ..., \Psi_t\in C[D]^*_{\leqslant M}$ be dual to $Q_1(D), ..., Q_t(D)$ which is to say that $\Psi_i(Q_j(D)) = \delta_{ij}$ for all i, j such that $1\leqslant i, j\leqslant t$. Set

$$f_i = \Phi(\Psi_i) \qquad \text{for} \quad i = 1, ..., t. \tag{4}$$

Since the Ψ_i are linearly independent in $C[D]^*_{\leqslant M}$ the f_i form a basis for $C[x]_{\leqslant M}$. Now $\langle f_i(x), Q_j(D)\rangle_{\leqslant M} = \Psi_i(Q_j(D)) = \delta_{ij}$ implies that $f_1(x), ..., f_s(x)$ forms a basis for $C[x]_{\leqslant M}$. Q.E.D

LEMMA 2. *Let*

$$z = z(D, M) = \sum_{|m| \leqslant M} X^m \otimes D^m. \tag{5}$$

Then $\Phi(\Psi) = f(x)$ *if and only if* $(1 \otimes \Psi) z = f(x)$.

Proof. Let $f(x) = \sum_{|m| \leqslant M} a_m X^m \in C[x]_{\leqslant M}$. Then $(1 \otimes \Psi) z = \sum_{|m| \leqslant M} X^m \Psi(D^m) = \sum_{|m| \leqslant M} X^m \langle f(x), D^m \rangle_{\leqslant M} = \sum_{|m| \leqslant M} X^m \langle \sum_{|p| \leqslant M} a_p X^p, D^m \rangle_{\leqslant M} = f(x)$. Q.E.D

COROLLARY 3. *For* $Q_i(D)$ *in* (3), $f_i(x)$ *in* (4) *and* z *in* (5) *we have*

$$z = \sum_{i=1}^{t} f_i(x) \otimes Q_i(D). \tag{6}$$

Proof. By a change of basis in (5) we can find $h_i(x)$ so that $z = \sum_{i=1}^{t} h_i(x) \otimes Q_i(D)$. Hence $f_j(x) = (1 \otimes \Psi_j) z = \sum_{i=1}^{t} h_i(x)\ \Psi_j(Q_i(D)) = h_j(x)$. Q.E.D

The point of Corollary 3 is that for any pair of dual bases $\{f_i(x)\}$ and $\{Q_i(D)\}$ we have (6).

THEOREM 4. *Let* β_i *represent the class of* $\partial/\partial x_i$ *in the residue class ring* $C[\partial/\partial x_1, ..., \partial/\partial x_n]/\langle P_1(\partial/\partial x_1, ..., \partial/\partial x_n), ..., P_v(\partial/\partial x_1, ..., \partial/\partial x_n)\rangle$. *Let* $C[D]_{\leqslant M}$ *have basis* $\{Q_i(D) \mid i = 1, ..., t\}$ *where* $\{Q_i(\beta) = 0 \mid i = s+1, ..., t\}$. *Then there exist* $f_i(x)$ *so that* $z(\beta, M) = \sum_{i=1}^{s} f_i(x) \otimes Q_i(\beta)$ *and* $\{f_i(x) \mid i = 1, ..., s\}$ *forms a basis for* $N_{\leqslant M}$.

Proof. By definition of the β_i we have that $P_j(\beta) = 0$ for $j = 1, 2, ..., v$, that $Q_j(\beta) = 0$ for $j = s+1, ..., t$ and that $Q_1(\beta), ..., Q_s(\beta)$ forms a basis for $C[\beta]_{\leqslant M}$. We apply vector space homomorphism $C[x] \otimes C[D] \to C[x] \otimes C[\beta]$ (where $D_j \to \beta_j$) to $z(D, M)$ to get $z(\beta, M) = \sum_{i=1}^{t} f_i(x) \otimes Q_i(\beta) = \sum_{i=1}^{s} f_i(x) \otimes Q_i(\beta)$. The proof of Lemma 1 shows that $\{f_i(x) \mid i = 1, ..., s\}$ forms a basis for $N_{\leqslant M}$. Q.E.D

Remarks. (1) The preceeding arguments work if we are finding real solutions to a system of real coefficient PDE's.

(2) There always exist subsets I of $\mathbf{N}_M^n$ so that $\{\beta^m \mid m \in I\}$ is a basis for $C[\beta]_{\leqslant M}$.

(3) It is easy to verify that $z(\beta, M) = \sum_{j=0}^{n} (\sum_{i=1}^{M} x_i \otimes \beta_i)^j / j!$.

(4) If $J_{\leqslant M} = \langle P_1(D), ..., P_v(D)\rangle$ contains 1 then $N_{\leqslant M} = \{0\}$ and there are no polynomial solutions.

(5) If the $P_k(D)$, $k=1, \ldots, r$ are of homogeneous order then $N=\bigoplus_{M=0}^{\infty} N_M$ and there is a basis of N consisting of homogeneous elements. We can find a basis for N_M by finding a basis for $C[\beta]_M$ and rewriting $(\sum_{i=1}^{n} x_i \otimes \beta_i)^M$ in terms of this basis. So if $C[\beta]_M$ has basis $Q_1(\beta), \ldots, Q_s(\beta)$ then there exist $f_1(x), \ldots, f_s(x)$ (homogeneous, degree M) so that $(\sum_{i=1}^{n} x_i \otimes \beta_i)^M = \sum_{i=1}^{s} f_i(x) \otimes Q_i(\beta)$ and we may conclude that $f_1(x), \ldots, f_s(x)$ forms a basis for N_M.

Examples. (1) Let $P(D)=(\partial^2/\partial x_1^2+\partial^2/\partial x_2^2)$. Let $R[\partial/\partial x_1, \partial/\partial x_2]/\langle P(D), D_1^{M+1}, D_1^M D_2, \ldots, D_2^{M+1}\rangle \cong R[\beta_1, \beta_2]$. Therefore $\beta_1^2+\beta_2^2=0$ and it is not hard to verify that $R[\beta]_{\leqslant M}$ has basis $\{\beta_1^k, \beta_1^{k-1}\beta_2 \mid 0 \leqslant k \leqslant M\}$. Now we can find $f_{k,1}(x)$, $f_{k,2}(x)$, $k=0, 1, \ldots, M$ so that $z(\beta, M)= \sum_{k=0}^{M} f_{k,1}(x) \otimes \beta_1^k + f_{k,2}(x) \otimes \beta_1^{k-1}\beta_2$. Theorem 4 implies that $f_{k,1}(x)$, $f_{k,2}(x)$, $k=0, 1, \ldots, M$ form a basis for the set of all polynomial solutions to Laplace's equation having degree $\leqslant M$. These solutions are in fact the same as the solutions one gets by finding the real and imaginary parts of the functions $(x_1+x_2 i)^k$, $k=0, \ldots, M$. If we use Remark 5 in this example, we have that $R[\beta]_M$ has basis $\{\beta_1^M, \beta_1^{M-1}\beta_2\}$. We then get that $(x_1 \otimes \beta_1 + x_2 \otimes \beta_2)^M = f_{M,1}(x) \otimes \beta_1^M + f_{M,2}(x) \otimes \beta_1^{M-1}\beta_2$ and we may conclude that $f_{M,1}(x)$ and $f_{M,2}(x)$ form a basis for N_M.

(2) Let $P(D)=(\partial^2/\partial x_1^2-\partial/\partial x_2)$ and let $R[\partial/\partial x_1^2, \partial/\partial x_2]/\langle P(D), D_1^4, D_1^3 D_2, \ldots, D_2^4\rangle \cong R[\beta_1, \beta_2]$ where $\beta_1^2-\beta_2=0$ and $\{\beta^m=0 \mid |m|=4\}$. Now $R[\beta]_{\leqslant 3}$ has basis 1, β_1, β_2, $\beta_1\beta_2$ and using this basis we have that $z(\beta, 3) = 1 \otimes 1 + x_1 \otimes \beta_1 + (x_2+x_1^2/2) \otimes \beta_2 + (x_1 x_2 + x_1^3/6) \otimes \beta_1\beta_2$. We conclude that the four coefficients 1, x_1, $(x_2+x_1^2/2)$, $(x_1x_2+x_1^3/6)$ form a basis for all real polynomial solutions to the heat equation having degree $\leqslant 3$.

(3) Let $P_1(D)=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2+\partial^2/\partial x_3^2$ and $P_2(D)=(\partial/\partial x_1)(\partial/\partial x_2)+(\partial/\partial x_1)(\partial/\partial x_3)+(\partial/\partial x_2)(\partial/\partial x_3)$. We will find a basis for the complex vector space N_3 by using Remark 5. To find these solutions we form the algebra $C[\beta]$ generated by β_i, $i=1, 2, 3$ where $\beta_1^2+\beta_2^2+\beta_3^2=0$, $\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3=0$ and $\{\beta^m=0 \mid |m|=4\}$. Now $C[\beta]_3$ has basis β_1^3, $\beta_1^2\beta_2$, $\beta_1^2\beta_3$, $\beta_1\beta_2^2$. We leave it to the reader to verify that these elements are indeed linearly independent in $C[\beta]_3$. The following table shows how the other elements of $C[\beta]_3$ can be written in terms of these elements.

$$\beta_1\beta_3^2 = -\beta_1^3 - \beta_1\beta_2^2, \qquad \beta_1\beta_2\beta_3 = -\beta_1^2\beta_2 - \beta_1^2\beta_3,$$

$$\beta_2^3 = -\beta_1^3 - 2\beta_1^2\beta_2 - \beta_1^2\beta_3 - \beta_1\beta_2^2, \qquad \beta_2^2\beta_3 = \beta_1^2\beta_2 + \beta_1^2\beta_3 - \beta_1\beta_2^2,$$

$$\beta_2\beta_3^2 = \beta_1^3 + \beta_1^2\beta_2 + \beta_1^2\beta_3 + \beta_1\beta_2^2, \qquad \beta_3^3 = -\beta_1^2\beta_2 - 2\beta_1^2\beta_3 + \beta_1\beta_2^2.$$

In $(\sum_{i=1}^{3} x_i \otimes \beta_i)^3$ we replace the β^m by the appropriate linear combination of our basis elements as given in the table. Collecting terms we get

$$\begin{aligned}\left(\sum_{i=1}^{3} x_i \otimes \beta_i\right)^3 &= (x_1^3/6 - x_1 x_3^2/2 - x_2^3/6 + x_2 x_3^2/2) \otimes \beta_1^3 \\ &\quad + (x_1^2 x_2/2 - x_1 x_2 x_3 - x_2^3/3 + x_2^2 x_3/2 + x_2 x_3^2/2 - x_3^3/6) \otimes \beta_1^2 \beta_2 \\ &\quad + (x_1^2 x_3/2 - x_1 x_2 x_3 - x_2^3/6 + x_2^2 x_3/2 + x_2 x_3^2/2 - x_3^3/3) \otimes \beta_1^2 \beta_3 \\ &\quad + (x_1 x_2^2/2 - x_1 x_3^2/2 - x_2^3/6 - x_2^2 x_3/2 + x_2 x_3^2/2 + x_3^3/6) \otimes \beta_1 \beta_2^2.\end{aligned}$$

Theorem 4 tells us that these four coefficients of the basis elements form a basis for N_3.

The Hilbert Characteristic Function

If $P_1(\partial/\partial x_1, ..., \partial/\partial x_n), ..., P_r(\partial/\partial x_1, ..., \partial/\partial x_n)$ are of homogeneous order, then $J = \langle P_1(\partial/\partial x_1, ..., \partial/\partial x_n), ..., P_r(\partial/\partial x_1, ..., \partial/\partial x_n) \rangle$ is a homogeneous ideal and the number of linearly independent forms of degree M in the residue class ring $C[\partial/\partial x_1, ..., \partial/\partial x_n]/J$ is designated $\chi(J; M)$. The function χ is called the Hilbert characteristic function of the ideal. A consequence of Lemma 1 is that $\chi(J; M) = \dim N_M$. Other papers have noted that this equality holds for sufficiently large M (see [4]). As a corollary to this observation we have that $\dim N_M$ is a polynomial in M for sufficiently large M (see, for example, [7, pp. 230–237]).

Polynomial and $\mathbf{C}^\infty$ Solutions to Systems

References [8, 9, and 1] discuss the relationship between polynomial solutions and $\mathbf{C}^\infty$ solutions to systems of constant coefficient PDE's. [10] and [9] discuss how one may find a basis for the $\mathbf{C}^\infty$ solutions to the system $\{u \in \mathbf{C}^\infty \mid P_k(\partial/\partial x_1, ..., \partial/\partial x_n)\ u = 0;\ k = 1, 2, ..., r\}$ by using functions of the form $Q(x) e^{\sum_{i=0}^{n} x_i \alpha_i}$ where $\{\alpha \in C^n \mid P_k(\alpha) = 0, k = 1, 2, ..., r\}$ and where the $Q(x)$ are certain unspecified polynomials. However this later work does not allow one to find a polynomial basis for the system.

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